Feb 15, 2023
Week 6

2020B Adv. Cal. II

Mass, Center of Mass, Centroid,
first moments $M_{xy}, M_{yz}, M_{xz}, M_x, M_y, M_z$,
moments of inertia I_x, I_y, I_z, I_o, etc

see Text.

Spherical coordinates

A point $P(x,y,z)$ in space can be described in (ρ, φ, θ) where

$$x = \rho \sin\varphi \cos\theta,$$
$$y = \rho \sin\varphi \sin\theta,$$
$$z = \rho \cos\varphi, \quad \rho \geq 0, \ \theta \in [0, 2\pi], \ \varphi \in [0, \pi].$$

(ρ, φ, θ) is the spherical coordinate of P. The map $(\rho, \varphi, \theta) \mapsto (x,y,z)$ maps $[0,\infty) \times [0,\pi] \times [0, 2\pi]$ onto $\mathbb{R}^3$, and is 1-1 onto (except the origin) when restricted to $(0,\infty) \times [0,\pi] \times [0, 2\pi)$.

When $\Omega \subset \mathbb{R}^3$ is described as

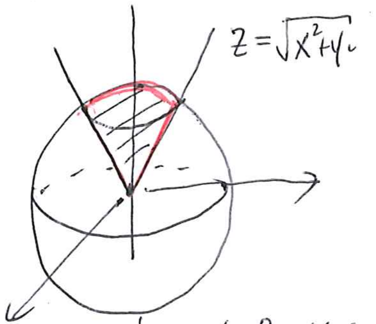
$$\{(x,y,z) : \rho_1(\varphi,\theta) \leq \rho \leq \rho_2(\varphi,\theta), \ \varphi_0 \leq \varphi \leq \varphi_1, \ \theta_0 \leq \theta \leq \theta_1\},$$

we have

$$\iiint_\Omega f \, dV = \int_{\theta_0}^{\theta_1} \int_{\varphi_0}^{\varphi_1} \int_{\rho_1(\varphi,\theta)}^{\rho_2(\varphi,\theta)} \hat{f}(\rho,\varphi,\theta) \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta,$$

where $\hat{f}(\rho,\varphi,\theta) \equiv f(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\theta)$.

e.g. Let Ω be the ice-cream cone bounded above by $x^2+y^2+z^2=4$ and below by $z=\sqrt{x^2+y^2}$. Find its volume and moment of inertia w.r.t. z-axis.



Express things in spherical coordinates:

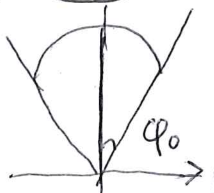
$$x^2+y^2+z^2=4,$$

$$(\rho\sin\varphi\cos\theta)^2+(\rho\sin\varphi\sin\theta)^2+(\rho\cos\varphi)^2=4,$$

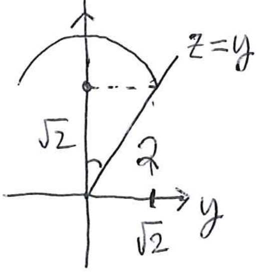
$$\rho^2=4$$

$$\rho=2 \qquad (\rho_1(\varphi,\theta)\equiv 2)$$

Clearly $\rho_0(\varphi,\theta)=0$.



Cross section on the yz-plane $(x=0)$



$x^2+y^2+z^2=4$ and $z=\sqrt{x^2+y^2}$ meets at:

$x^2+y^2+x^2+y^2=4$, $x^2+y^2=2$, and

$z=\sqrt{x^2+y^2}=\sqrt{2}$.

$$\tan\varphi_0=\frac{\sqrt{2}}{\sqrt{2}} \Rightarrow \varphi_0=\pi/4.$$

$$\therefore\ \Omega:\quad 0\le\rho\le 2,\ 0\le\varphi\le\pi/4,\ 0\le\theta\le 2\pi.$$

$$\text{vol}=\iiint_\Omega 1\,dV=\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$\vdots$$

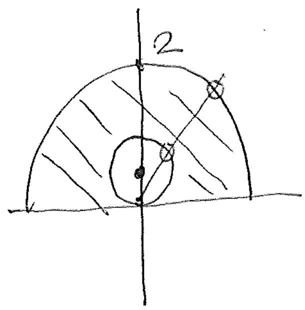
$$=\frac{8}{3}\left(1-\frac{\sqrt{2}}{2}\right)2\pi.$$

$$I_z=\iiint_\Omega (x^2+y^2)\,dV=\int_0^{2\pi}\int_0^{\pi/4}\int_0^2\left[(\rho\sin\varphi\cos\theta)^2+(\rho\sin\varphi\sin\varphi)^2\right]\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$\vdots$$

$$=(8-3\sqrt{2})\frac{16\pi}{15}\quad \#$$

e.g. Let Ω be the solid bdd between $x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4}$ and the hemisphere $x^2+y^2+z^2=4$, $z\geq 0$. Find its volume.



a cross section of Ω.

ρ_1 described by $x^2+y^2+z^2=4$, $\therefore$ $\rho_1(\varphi,\theta)\equiv 2$.

ρ_0 described by $x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4}$, ie

$$(\rho\sin\varphi\cos\theta)^2+(\rho\sin\varphi\sin\theta)^2+(\rho\cos\ -\tfrac{1}{2})^2=\tfrac{1}{4},$$

$$\rho^2\sin^2\varphi+\rho^2\cos^2\varphi-\rho\cos\varphi+\tfrac{1}{4}=\tfrac{1}{4},$$

$$\rho^2-\rho\cos\varphi=0,$$

$$\rho=\cos\varphi$$

$\therefore$ $\rho_0(\varphi,\theta)=\cos\varphi$. $\qquad$ $\therefore$ Ω is

$$\cos\varphi\leq\rho\leq 2$$
$$0\leq\varphi\leq\pi/2$$
$$0\leq\theta\leq 2\pi.$$

$$\text{vol}=\int_0^{2\pi}\int_0^{\pi/2}\int_{\cos\varphi}^{2} 1\ \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$\vdots$

$$=\frac{31\pi}{6}\ \#$$

A point $P(x,y,z)$ can be described by (r,θ,z) where $x=r\cos\theta$, $y=r\sin\theta$, $z=z$. This is called the cylindrical coordinates of P.

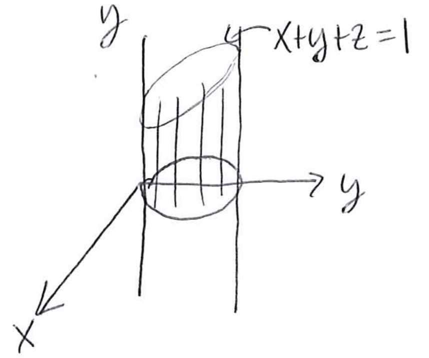
When $\Omega\subset\mathbb{R}^3$ is described as

$$g_1(r,\theta)\leq z\leq g_2(r,\theta)$$
$$(r,\theta)\in D,$$

we have

$$\iiint_\Omega f\,dV=\iint_D\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,dz\,r\,dr\,d\theta.$$

e.g. Let Ω be the solid bdd above by $x+y+z=4$, below by $z=0$, and by $x^2+(y-1)^2=1$ on the side. Find its volume.



Use cylindrical coordinates,

$$x^2+(y-1)^2=1$$
$$(r\cos\theta)^2+(r\sin\theta-1)^2=1$$
$$r^2\cos^2\theta+r^2\sin^2\theta-2r\sin\theta+1=1$$
$$r=2\sin\theta.$$

$$\Omega: \quad 0\le z\le 4-x-y=4-r\cos\theta-r\sin\theta.$$
$$0\le\theta\le 2\pi$$
$$0\le r\le 2\sin\theta.$$

$$\text{vol}=\iiint_\Omega dV=\int_0^{2\pi}\int_0^{2\sin\theta}\int_0^{4-r\cos\theta-r\sin\theta} dz\,r\,dr\,d\theta$$

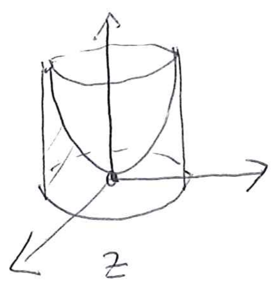
$$=\int_0^{2\pi}\int_0^{2\sin\theta}(4-r\cos\theta-r\sin\theta)\,r\,dr\,d\theta$$

$$\vdots$$

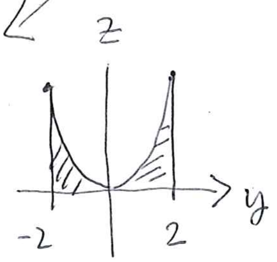
$$=6\pi.$$

e.g. Let Ω be the region bdd above by $z=x^2+y^2$, below by the xy-plane, and the cylinder $x^2+y^2=4$ on the side. Find its centroid.

see below



By symmetry, $\bar{x}=\bar{y}=0$, need to find $\bar{z}$.



$$\Omega: \quad 0\le z\le x^2+y^2=r^2$$
$$0\le\theta\le 2\pi$$
$$0\le r\le 2$$

$$M=\iiint_\Omega 1\,dV=\int_0^{2\pi}\int_0^2\int_0^{r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 z\Big|_0^{r^2} r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^4}{4}\Big|_0^2 d\theta = 8\pi .$$

$$M_{xy} = \int_0^{2\pi}\int_0^2\int_0^{r^2} z\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{1}{2}z^2\Big|_0^{r^2} r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \frac{1}{2}r^5\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\frac{1}{6}r^6\Big|_0^2 d\theta = \frac{16}{3}\int_0^{2\pi} d\theta = \frac{32}{3}\pi .$$

$$\therefore \bar{z} = \frac{32}{3}\pi \Big/ 8\pi = \frac{4}{3} .$$

$$C = (\bar{x}, \bar{y}, \bar{z}) = (0, 0, \tfrac{4}{3}) . \ \#$$

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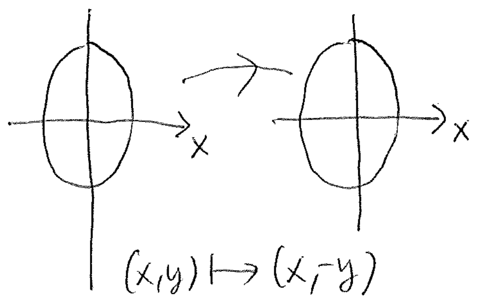
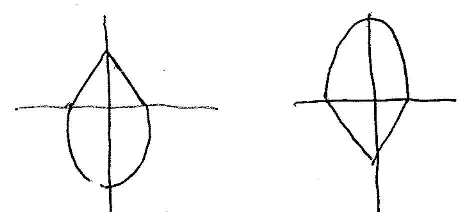
A symmetry result

$D \subset \mathbb{R}^2$ is symmetric about the reflection in the x-axis if $D = \tilde{D}$ where $\{(x,y) : (x,-y) \in D\} \equiv \tilde{D}$



$(x,y) \mapsto (x,-y)$
the same after reflection



$(x,y) \mapsto (x,-y)$
not the same after reflection.

When f is defined on D which is symmetric about the x-axis, $f(x,-y)$ is defined in $\tilde{D} = D$, ie, $f(x,-y)$ and $f(x,y)$ are defined in D.

Theorem Let D be symmetric about the x-axis and f satisfies

$$f(x,-y) = -f(x,y) \quad (\text{odd in } y)$$

$$\iint_D f(x,y)\,dA = 0.$$

Pf. Let $\tilde{f}$ be the universal extension of f. f odd in $y \Rightarrow$ $\tilde{f}$ also odd in y. Let $D \subset [-a,a]\times[-c,c]$.

$$\iint_D f(x,y)\,dA \equiv \int_{-a}^{a}\int_{-c}^{c} \tilde{f}(x,y)\,dy\,dx \text{ , by Fubini's theorem.}$$

As

$$\int_{-c}^{c} \tilde{f}(x,y)\,dy = 0 \qquad \left(\text{recall } f=f(t) \text{ odd fcn, } \int_{-a}^{a} f(t)\,dt = 0\right)$$

$$\therefore \iint_D f(x,y)\,dA = 0 . \quad \square$$

Similarly, if D is symmetric about the y-axis and f satisfies $f(-x,y) = -f(x,y)$, then

$$\iint_D f(x,y)\,dA = 0.$$

In space, Ω is symmetric w.r.t xy-plane if Ω is unchanged under $(x,y,z) \mapsto (x,y,-z)$.

When f is odd in z, ie, $f(x,y,-z) = -f(x,y,z)$, we have

$$\iiint_\Omega f\,dV = 0.$$

Similarly, when Ω is symmetric w.r.t. xz-plane and f satisfies $f(x,-y,z) = -f(x,y,z)$,

$$\iiint_{\Omega} f\,dV = 0.$$

When Ω is symmetric w.r.t. yz-plane and f satisfies $f(-x,y,z) = -f(x,y,z)$, then

$$\iiint_{\Omega} f\,dV = 0.$$

In our previous example, Ω is bdd above by $z = x^2+y^2$ below by xy-plane, and the cylinder $x^2+y^2=4$ on the side.

Clearly, Ω is symmetric w.r.t. yz-plane and xz-plane.

Hence $\iiint_{\Omega} x\,dV = \iiint_{\Omega} y\,dV = 0$, as

$f(x,y,z) = x$ is odd in x and $g(x,y,z) = y$ is odd in y.